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STIFF SYSTEMS OF HYPERBOLIC CONSERVATION LAWS: CONVERGENCE AND ERROR ESTIMATES*

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Abstract. We are concerned with 2×2 nonlinear relaxation systems of conservation laws of the form $u_t + f(u)_x = -\frac{1}{\delta}S(u,v)$, $v_t = \frac{1}{\delta}S(u,v)$ which are coupled through the stiff source term $\frac{1}{\delta}S(u,v)$. Such systems arise as prototype models for combustion, adsorption, etc. Here we study the convergence of $(u,v) \equiv (u^\delta, v^\delta)$ to its equilibrium state, $(\bar{u},\bar{v})$, governed by the limiting equations, $\bar{u}_t + \bar{v}_t + f(\bar{u})_x = 0$, $S(\bar{u},\bar{v}) = 0$. In particular, we provide sharp convergence rate estimates as the relaxation parameter $\delta \downarrow 0$. The novelty of our approach is the use of a weak $W^{-1}(L^1)$-measure of the error, which allows us to obtain sharp error estimates. It is shown that the error consists of an initial contribution of size $||S(u_0^\delta, v_0^\delta)||_{L^1}$, together with accumulated relaxation error of order $\mathcal{O}(\delta)$. The sharpness of our results is found to be in complete agreement with the numerical experiments reported in [Schroll, Tveito, and Winther, *SIAM J. Numer. Anal.*, 34 (1997), pp. 1152–1166].

Key words. conservation laws, stiff source terms, relaxation, *Lip*$^+$-stability, convergence rate estimates

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1. Introduction.

We are concerned with one-dimensional systems of conservation laws which are coupled through a stiff source term. The purpose of this paper is to study a convergence rate of such systems to their equilibrium solutions as the stiff relaxation parameter tends to zero.

Our system takes the form

$$u_t + f(u)_x = -\frac{1}{\delta}S(u,v), \tag{1.1}$$

$$v_t = \frac{1}{\delta}S(u,v), \tag{1.2}$$

where $\delta > 0$ is the small relaxation parameter. The stiff source term, $S(u,v)$, and the convective flux, $f(u)$, are assumed to be smooth functions. We consider the Cauchy problem associated with (1.1)–(1.2), subject to periodic or compactly supported initial data

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x). \tag{1.3}$$

Here $u(x,t) := u^\delta(x,t)$, $v(x,t) := v^\delta(x,t)$ is the unique entropy solution of (1.1)–(1.3), which can be realized as the vanishing viscosity limit $u^\delta = \lim_{\nu\downarrow 0} u^{\delta,\nu}$, $v^\delta = \lim_{\nu\downarrow 0} v^{\delta,\nu}$, where $(u^{\delta,\nu}, v^{\delta,\nu})$ is the solution of the regularized viscosity system

$$u_t^{\delta,\nu} + f(u^{\delta,\nu})_x = -\frac{1}{\delta}S(u^{\delta,\nu}, v^{\delta,\nu}) + \nu u_{xx}^{\delta,\nu}, \tag{1.4}$$

$$v_t^{\delta,\nu} = \frac{1}{\delta}S(u^{\delta,\nu}, v^{\delta,\nu}). \tag{1.5}$$

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This regularized system, with fixed $\delta > 0$ (and $\nu > 0$), admits a unique, global (and, respectively, classical) solution. Indeed, such a solution can be constructed, for example, by fixed point iterations which alternate between the solution of the ODE (1.5) for v and the viscous conservation law—with v-dependent source term (1.4) for u. Moreover, by the maximum principle, e.g., [PW], the solution constructed admits a global uniform bound in view of our monotonicity assumption specified in section 2, $-S_u, S_v \leq 0$. Finally, by standard arguments (which we omit), there exists a constant, *independent* of ν, $C_\delta = \exp\{2(|S_u| + |S_v|)t/\delta\}$, such that

$$\begin{aligned}&\|u_1^\delta(\cdot,t) - u_2^\delta(\cdot,t)\|_{L^1} + \|v_1^\delta(\cdot,t) - v_2^\delta(\cdot,t)\|_{L^1}\\ &\quad \leq C_\delta\left[\|u_1^\delta(\cdot,0) - u_2^\delta(\cdot,0)\|_{L^1} + \|v_1^\delta(\cdot,0) - v_2^\delta(\cdot,0)\|_{L^1}\right].\end{aligned}$$

Consequently, the uniqueness of the viscous solution, $u^{\delta,\nu}$, and hence the uniqueness of its entropy limit BV-solution, u^δ, then follow. We refer to, e.g., [HW], [Lu], and [Le] for further discussions on the existence and uniqueness for various related models of the above type.

Once we identify the unique entropy solution, (u^δ, v^δ), we seek its equilibrium state as $\delta \downarrow 0$, $(\bar{u}, \bar{v})$. Formally, our equilibrium solution is governed by the limit system obtained by letting $\delta \downarrow 0$ in (1.1)–(1.2),

$$(\bar{u} + \bar{v})_t + f(\bar{u})_x = 0, \tag{1.6}$$

$$S(\bar{u}, \bar{v}) = 0. \tag{1.7}$$

To obtain the limiting equation (1.6), add (1.2) to (1.1); to obtain the constraint equation (1.7), multiply (1.2) by δ and pass to the formal limit as $\delta \to 0$.

The two main questions that we address in this paper are concerned with the convergence of the entropy solution (u^δ, v^δ) to its expected equilibrium state $(\bar{u}, \bar{v})$.

Convergence. We prove the convergence to the expected limits

$$\bar{u} = \lim_{\delta,\nu\downarrow 0} u^{\delta,\nu}, \qquad \bar{v} = \lim_{\delta,\nu\downarrow 0} v^{\delta,\nu}. \tag{1.8}$$

Moreover, we provide the following.

Error estimates. We estimate the convergence rate as $\nu \to 0$ and, in particular, as $\delta \to 0$.

Assume that $S_v \neq 0$ so we can solve the constraint equation (1.7) and obtain its solution in the explicit form

$$\bar{v} = v(\bar{u}). \tag{1.9}$$

Inserted into (1.6), we obtain that $\bar{u}$ is governed by the limiting equation

$$[\bar{u} + v(\bar{u})]_t + f(\bar{u})_x = 0. \tag{1.10}$$

Equivalently, if we denote $\bar{w} = \bar{w}(\bar{u}) := \bar{u} + v(\bar{u})$ and let its inverse[1] $\bar{u} = \bar{u}(\bar{w})$, then we conclude that the limiting equation (1.10) can be rewritten as a single conservation law, expressed in terms of the combined flux $F(\bar{w}) := f(\bar{u}(\bar{w}))$,

$$\bar{w}_t + F(\bar{w})_x = 0. \tag{1.11}$$

[1] The inverse exists since by our monotonicity assumption in section 2 below, $v'(u) = -S_u/S_v > -1$.

We obtain our convergence results under the assumptions of convexity—both $f(\cdot)$ and $F(\cdot)$ and the monotonicity of $S(u,v)$. In addition, we assume that we start with "prepared" initial data, in the sense that $u_0 \equiv u_0^\delta$ and $v_0 \equiv v_0^\delta$ approach their equilibrium state (1.7) as $\delta \downarrow 0$, i.e.,

$$||S(u_0^\delta(x), v_0^\delta(x))||_{L^1(x)} \stackrel{\delta\to 0}{\longrightarrow} 0.$$

Specifically, we let $\epsilon = \epsilon(\delta) \downarrow 0$ denote the vanishing *initial error*

$$||S(u_0^\delta(x), v_0^\delta(x))||_{L^1(x)} \sim \epsilon(\delta) \downarrow 0. \tag{1.12}$$

Equipped with these assumptions, we formulate in section 2 our main results, which we summarize here in the following theorem.

THEOREM 1.1 (main theorem). *Consider the system* (1.3)–(1.5) *subject to* $W^2(L^1)$-*"prepared" initial data,* (1.12). *Then* $(u^{\delta,\nu}, v^{\delta,\nu})$ *converges to* $(\bar{u}, \bar{v})$ *as* $\nu \to 0$, $\delta \to 0$, *and the following error estimate holds* $\forall p$, $1 \le p \le \infty$*:*

$$||u^{\delta,\nu}(\cdot,t) - \bar{u}(\cdot,t)||_{W^s(L^p(x))} \le \mathrm{Const}_T \cdot \big(\epsilon(\delta) + \delta + \nu\big)^{\frac{1-sp}{2p}}, \quad -1 \le s \le \frac{1}{p}\ . \tag{1.13}$$

Thus, (1.13) reflects three sources for error accumulation: the initial error of size $\epsilon(\delta)$, the relaxation error of order δ, and the vanishing viscosity of order ν. For example, in the inviscid case ($\nu = 0$) and with "canonically prepared" initial data such that $\epsilon(\delta) \sim \delta$, we set $(s,p) = (0,1)$ in (1.13) to conclude an L^1-convergence rate of order $\mathcal{O}(\sqrt{\delta})$; in fact, in Corollary 2.3 below we extend this L^1-estimate to the v-variable, stating that

$$||u^\delta(\cdot,t) - \bar{u}(\cdot,t)||_{L^1} + \|v^\delta(\cdot,t) - \bar{v}(\cdot,t)\|_{L^1} = \mathcal{O}(\sqrt{\delta}). \tag{1.14}$$

The two-step proof of the main theorem is presented in sections 3 (stability) and 4 (consistency).

We close this introduction with three prototype examples.

Example 1: *Combustion.* We consider a combustion model proposed by Majda [Ma]. This model was consequently studied in [Le], [TY], and [Lu]. It takes the form

$$\begin{aligned} u_t + f(u)_x &= \frac{1}{\delta} A(u)v + \nu u_{xx}, \\ v_t &= -\frac{1}{\delta} A(u)v. \end{aligned} \tag{1.15}$$

Here $u \equiv u^{\delta,\nu}$ is a lumped variable representing some features of density, velocity, and temperature, while $v \equiv v^{\delta,\nu} \ge 0$ represents the mass fraction of unburnt gas in a simplified kinetics scheme; $\frac{1}{\delta}$ is the rate of reaction and the parameter $\nu > 0$ is a lumped parameter representing the effects of diffusion and heat conduction.

In this model, $S(u,v) = -A(u)v$ and our convexity and monotonicity assumptions (2.1)–(2.3) below hold, provided that

$$A'(u) < 0, \ A(u) \ge \eta > 0; \quad f''(u) \ge \alpha > 0. \tag{1.16}$$

The limiting equation (1.10) in this example reads

$$\bar{u}_t + f(\bar{u})_x = 0,$$

and hence $u^{\delta,\nu} - \bar{u}$ satisfies the error estimate (1.13).

Example 2: *Adsorption.* We consider the following stiff system:

$$\begin{aligned} u_t + f(u)_x &= -\frac{1}{\delta}(A(u) - v), \\ v_t &= \frac{1}{\delta}(A(u) - v). \end{aligned} \tag{1.17}$$

In this example $u \equiv u^\delta$ denotes the density of some species contained in a fluid flowing through a fixed bed, and $v \equiv v^\delta$ denotes the density of the species adsorbed on the material in the bed; $\delta > 0$ is referred to as the relaxation time. Different forms of adsorption functions, $A(u)$, are discussed in [STW], [TW1], [TW2], and the references therein.

The source term associated with this adsorption model, $S(u,v) = A(u)-v$, yields a limiting equation of the form

$$[\bar{u} + A(\bar{u})]_t + f(\bar{u})_x = 0.$$

Under the monotonicity assumption and convexity condition (consult (2.1)–(2.3)),

$$A'(u) \geq 0, \quad \left[\frac{f'(u)}{1 + A'(u)}\right]' \geq \alpha > 0. \tag{1.18}$$

We conclude the error estimate (1.13) with $\nu = 0$. In particular, for "canonically prepared" initial data such that $||A(u_0^\delta) - v_0^\delta||_{L^1} = \mathcal{O}(\delta)$, (1.14) yields a convergence rate of order $\mathcal{O}(\sqrt{\delta})$.

In this context it is interesting to contrast our above error estimates with those of [STW]. In [STW], Schroll, Tveito, and Winther studied the error estimates for the adsorption model (1.17) subject to "canonically prepared" initial data, $||A(u_0^\delta) - v_0^\delta||_{L^1} = \mathcal{O}(\delta)$, and concluded an L^1-convergence rate of order $\mathcal{O}(\delta^{\frac{1}{3}})$. Their reported numerical experiments, however, indicate a faster convergence rate of order $\mathcal{O}(\sqrt{\delta})$. Our results, e.g., (1.14), apply to their numerical experiments and confirm this optimal $\mathcal{O}(\sqrt{\delta})$ convergence rate. It should be pointed that the $\mathcal{O}(\delta^{\frac{1}{3}})$ error estimate in [STW] was derived by interpolation between L^2- and L^1-error bounds. It is here that we take advantage of our *sharper* interpolation between the *weaker* $\mathcal{O}(\delta)$ Lip'- and the $\mathcal{O}(1)$ BV-bounds. This enables us to improve over [STW] in both simplicity and generality and conclude with the sharper estimate of order $\mathcal{O}(\sqrt{\delta})$.

Example 3: *Relaxation.* Let us consider the following semilinear stiff system (see, e.g., [JX], [Li]):

$$\begin{aligned} u_t + v_x &= 0, \\ v_t + au_x &= \frac{1}{\delta}S(u,v), \end{aligned} \tag{1.19}$$

where $S(u,v) := f(u) - v$ and a is given positive number. The limiting equation, with $v(u) = f(u)$, is then

$$\bar{u}_t + f(\bar{u})_x = 0.$$

To study this system we rewrite it in the form of (1.1)–(1.2) by means of two changes of variables. First, we define the characteristic variables $w := \sqrt{a}\,u + v$, $z :=$

$\sqrt{a}\,u - v$. The system (1.19) then takes the form

$$
\begin{aligned}
z_t - \sqrt{a}\, z_x &= -\frac{1}{\delta} S(z,w),\\
w_t + \sqrt{a}\, w_x &= \frac{1}{\delta} S(z,w),
\end{aligned}
\tag{1.20}
$$

with $S(z,w) = S(u(z,w), v(z,w))$. Next, we make the second change of variables, $x' := x - \sqrt{a}\, t$, obtaining

$$
\begin{aligned}
z_t - 2\sqrt{a}\, z_{x'} &= -\frac{1}{\delta} S(z,w),\\
w_t &= \frac{1}{\delta} S(z,w).
\end{aligned}
\tag{1.21}
$$

In this model, the flux is linear and hence our first convexity assumption, (2.2), holds. The second one, (2.3), is satisfied for convex f's. In addition, the monotonicity of S, $S_z \geq 0, S_w \leq -\eta < 0$, amounts (in terms of S_u and S_v) to the inequalities

$$S_v \leq -\eta < 0, \quad S_v\sqrt{a} \leq S_u \leq -S_v\sqrt{a}.$$

Thus, $S(u,v) = f(u) - v$ should satisfy Liu's *subcharacteristic* condition (e.g., [Li]),

$$-\sqrt{a} \leq f'(u) \leq \sqrt{a}.$$

In this case, our main theorem with $p = 1$, for example, yields

$$||u^\delta - \bar{u}||_{W^s(L^1)} = \text{Const} \cdot \left(||f(u_0^\delta) - v_0^\delta||_{L^1} + \delta\right)^{\frac{1-s}{2}}, \quad -1 \leq s \leq 1.$$

2. Statement of main results. We seek the behavior of the solution of regularized system (1.4)–(1.5) towards the limit solution as $\delta \to 0$, as well as $\nu \to 0$. Throughout this section we make the following two main assumptions.

Monotonicity. $S(u,v)$ is monotonic with respect to u and strictly monotonic with respect to v,

$$S_u(u,v) \geq 0, \quad S_v(u,v) \leq -\eta < 0. \tag{2.1}$$

Convexity. $f(\cdot)$ is convex and $F(\cdot)$ is a strictly convex function,

$$f''(u) \geq 0, \tag{2.2}$$

$$F''(w) \geq \alpha > 0 \Longleftrightarrow \left(\frac{f'(\bar{u})}{1 + v'(\bar{u})}\right)' \geq \alpha > 0. \tag{2.3}$$

Remark. Our first assumption of monotonicity guarantees, by the classical maximum principle (see, e.g., [PW]), the L^∞-boundedness of $(u^{\delta,\nu}, v^{\delta,\nu})$ (proof is left to the reader).

Equipped with the two assumptions above, we now turn to the main result of this paper. To this end, our error estimate is formulated in terms of the *weak Lip'*-(semi)norm, $||\cdot||_{Lip'}$, and the dual of the *Lip*-norm given by

$$||\phi||_{Lip'} := \sup_{\psi}[(\phi - \hat{\phi_0}, \psi)/||\psi||_{W^{1,\infty}}], \quad \hat{\phi_0} := \int_{supp\phi} \phi.$$

Thus, the Lip'-size of regular ϕ's (with bounded average over their finite support) amounts to their $W^{-1}(L^1)$-size or, equivalently, the L^1-size of their *primitive*. As we shall see, such weak (semi)norm has the advantage of providing us with *sharp* error estimates which, in turn, will be converted into strong ones.

THEOREM 2.1. *Consider the system* (1.3)–(1.5) *subject to* $W^2(L^1)$*-"prepared" initial data,* (1.12)*. Then* $(u^{\delta,\epsilon}, v^{\delta,\epsilon})$ *converges to* $(\bar{u}, \bar{v})$ *as* $\delta \to 0$, $\nu \to 0$, *and the following error estimate holds:*

$$||u^{\delta,\nu}(\cdot,T) - \bar{u}(\cdot,T)||_{Lip'(x)} \leq \text{Const}_T \cdot \big(\epsilon(\delta) + \delta + \nu\big). \tag{2.4}$$

Let us consider the particular inviscid case, where $\nu = 0$. Then the entropy solution of the stiff system (1.1)–(1.2), (u^δ, v^δ), converges as $\delta \to 0$ to its equilibrium solution, $(\bar{u}, \bar{v})$, and we obtain the asserted convergence rate in terms of the initial error $\epsilon(\delta)$ and the vanishing relaxation parameter δ:

$$||u^{\delta}(\cdot,T) - \bar{u}(\cdot,T)||_{Lip'(x)} \leq \text{Const}_T \cdot \big(\epsilon(\delta) + \delta\big). \tag{2.5}$$

Remarks. 1. Our assumption of "prepared" initial data means that at the initial moment, $||S(u_0^\delta, v_0^\delta)||_{L^1} \stackrel{\delta\to 0}{\longrightarrow} 0$. In section 4 we will show that, in fact, $||S(u^{\delta,\nu}, v^{\delta,\nu})||_{L^1} \stackrel{\delta\to 0}{\longrightarrow} 0$ for all $t > 0$.

2. What about "nonprepared" initial data? In this case the initial layer formed persists in time; i.e., the initial error propagates and prevents convergence of $u^{\delta,\nu}, v^{\delta,\nu}$ to their equilibrium state.

The proof of the main theorem will be given in sections 3 and 4. To obtain this result we utilize the framework of Tadmor and Nessyahu [Ta], [NT]. To this end, we need the two ingredients of *consistency* and *stability*. Here, *consistency*—evaluated in terms of the Lip'-norm—measures by how much the approximate pair $(u^{\delta,\nu}, v(u^{\delta,\nu}))$ fails to satisfy the limiting equation (1.10); *stability* requires the Lip^+-stability [2] of $u^{\delta,\nu}$; that is, we seek a one-sided Lipschitz continuity (OSLC) of the viscosity solution $u^{\delta,\nu}$,

$$\|u^{\delta,\nu}(\cdot,t)\|_{Lip^+(x)} := \sup_x\, [u_x^{\delta,\nu}(x,t)]_+ \leq C_t \cdot \|u^{\delta,\nu}(\cdot,0)\|_{Lip^+(x)}. \tag{2.6}$$

By interpolation between the (weak) Lip'-error estimate (2.4) and the (strong) BV-boundedness of the error (which follows from the Lip^+-boundedness due to (2.6)), we are able to convert the weak error estimate stated in Theorem 2.1 into a *strong* one. As in [NT], we conclude with the following corollary.

COROLLARY 2.2 (global estimate). *Consider the inviscid problem* (1.1)–(1.3)*,* (1.12)*. Then the following convergence rate estimate holds:*

$$||u^{\delta}(x,T) - \bar{u}(x,T)||_{L^p} \leq \text{Const}_T \cdot (\epsilon(\delta) + \delta)^{\frac{1}{2p}}, \qquad 1 \leq p \leq \infty. \tag{2.7}$$

Remark. The above-mentioned L^p-estimates in (2.7) are, in fact, particular cases of the more general error estimate in the $W^s(L^p)$-norm

$$||u^{\delta}(x,T) - \bar{u}(x,T)||_{W^s(L^p)} \leq \text{Const}_T \cdot (\epsilon(\delta) + \delta)^{\frac{1-sp}{2p}}, \qquad -1 \leq s \leq \frac{1}{p}\ . \tag{2.8}$$

[2] Here $||\phi||_{Lip^+} := ess\sup_{x\neq y} \left[\frac{\phi(x)-\phi(y)}{x-y}\right]_+$, where, as usual, $(\cdot)_+$ denotes the "positive part of." For convenience we shall use the equivalent definition of the Lip^+ norm: $||\phi||_{Lip^+} := \sup_x\, [\phi'(x)]_+$, where the derivative of ϕ is taken in the distribution sense.

The special cases, $(s,p)=(-1,1)$ and $s=0$, correspond, respectively, to the weak Lip'-estimate (Theorem 2.1) and the global L^p-estimate (Corollary 2.2).

Taking $p=1$ in (2.7), we obtain, in particular, the L^1-error estimate, which reads

$$||u^\delta(x,T)-\bar{u}(x,T)||_{L^1} \le \text{Const}_T \cdot \sqrt{\epsilon(\delta)+\delta}. \tag{2.9}$$

In this L^1-framework, we are able to extend the last estimate and obtain the same $\mathcal{O}(\sqrt{\epsilon(\delta)+\delta})$ convergence rate of v^δ towards $\bar{v}$. This brings us to the following corollary.

Corollary 2.3 (L^1-error estimate). *Consider the system* (1.1)–(1.3) *subject to "prepared" initial data,* (1.12). *Then we have*

$$||u^\delta(x,T)-\bar{u}(x,T)||_{L^1}+||v^\delta(x,T)-\bar{v}(x,T)||_{L^1} \le \text{Const}_T \cdot \sqrt{\epsilon(\delta)+\delta}. \tag{2.10}$$

In particular, for "canonically prepared" initial data, $||S(u_0^\delta,v_0^\delta)||_{L^1}=\epsilon(\delta)\sim\delta$, *we obtain a convergence rate of order* $\sqrt{\delta}$,

$$||u^\delta(x,T)-\bar{u}(x,T)||_{L^1}+||v^\delta(x,T)-\bar{v}(x,T)||_{L^1} \le \text{Const}_T \cdot \sqrt{\delta}. \tag{2.11}$$

Proof. We first note that due to the strict monotonicity of $S(u,v)$ with respect to its second argument and the L^∞-bound of $u^\delta, v^\delta, \bar{u}$, and $\bar{v}$, we have

$$\begin{aligned}|v^\delta-\bar{v}| &= |v^\delta - v(u^\delta)+v(u^\delta)-\bar{v}| \le |v^\delta - v(v^\delta)|+|v(v^\delta)-\bar{v}| \\ &= |v'(\tilde{u})|\cdot|u^\delta-\bar{u}| + \left|\frac{S(u^\delta,v^\delta)-S(u^\delta,v(u^\delta))}{S_v(u^\delta,\tilde{v})}\right| \sim |u^\delta-\bar{u}|+|S(u^\delta,v^\delta)|.\end{aligned}$$

Here $\tilde{u}$ and $\tilde{v}$ are appropriate midvalues, $\tilde{u}=\theta_1 u^\delta+(1-\theta_1)\bar{u}$, $\tilde{v}=\theta_2 v^\delta+(1-\theta_2)\bar{v}$. And we now obtain the desired estimate, (2.10),

$$\begin{aligned}||v^\delta(x,T)-\bar{v}(x,T)||_{L^1} &\le \text{Const}_T\cdot(||u^\delta(x,T)-\bar{u}(x,T)||_{L^1} \\ &\qquad +||S(u^\delta(x,T),v^\delta(x,T))||_{L^1}) \\ &= \mathcal{O}(\sqrt{\epsilon(\delta)+\delta})+\mathcal{O}(\epsilon(\delta)+\delta)=\mathcal{O}(\sqrt{\epsilon(\delta)+\delta}).\end{aligned} \tag{2.12}$$

Indeed, the first $\mathcal{O}(\sqrt{\bullet})$-upperbound on the right is due to (2.9); the second upperbound, $||S(u^\delta(x,T),v^\delta(x,T))||_{L^1}=\mathcal{O}(\epsilon(\delta)+\delta)$, is outlined in section 4 below. ◻

Finally, arguing along the lines of [NT; Corollary 2.4], we also obtain the *pointwise* convergence towards the equilibrium solution away from discontinuities.

Corollary 2.4 (local estimate). *Consider the inviscid problem* (1.1)–(1.3), (1.12). *Then the following estimate holds:*

$$|u^\delta(x,T)-\bar{u}(x,T)| \le \text{Const}_{x,T}\cdot(\epsilon(\delta)+\delta)^{\frac{1}{3}}. \tag{2.13}$$

Here, $\text{Const}_{x,T}$ *is a constant which measures the local smoothness of* $u(\cdot,T)$ *in the small neighborhood of* x,

$$\text{Const}_{x,T}\sim 1+\max_{|y-x|<\sqrt[3]{\delta}}|\bar{u}_x(y,T)|.$$

.

3. *Lip*$^+$-stability estimate. We now turn to the proof of our main theorem. We begin with the *Lip*$^+$-stability of the solution of (1.4)–(1.5).

ASSERTION 3.1. *Consider the system* (1.4), (1.5) *subject to Lip*$^+$*-bounded initial data* (1.3). *Then there exists a constant (which may depend on the initial data) such that*

$$||u^{\delta,\nu}(\cdot,T)||_{Lip^+(x)} \leq \text{Const.} \tag{3.1}$$

Proof. The proof is based on the maximum principle for $(u_x^{\delta,\nu})_+$.

Differentiation of (1.4) and (1.5) with respect to x implies

$$(u_x^{\delta,\nu})_t + f''(u^{\delta,\nu})(u_x^{\delta,\nu})^2 + f'(u^{\delta,\nu})(u_x^{\delta,\nu})_x = -\frac{1}{\delta}[S_u u_x^{\delta,\nu} + S_v v_x^{\delta,\nu}] + \nu(u_x^{\delta,\nu})_{xx}, \tag{3.2}$$

$$(v_x^{\delta,\nu})_t = \frac{1}{\delta}[S_u u_x^{\delta,\nu} + S_v v_x^{\delta,\nu}]. \tag{3.3}$$

We now multiply (3.2) by $\frac{1+sgn(u_x^{\delta,\nu})}{2}$; using the monotonicity of $S(u,v)$ and convexity of $f(u)$ we obtain the following inequalities:

$$\begin{aligned}&[(u_x^{\delta,\nu})_+]_t + f'(u^{\delta,\nu})\cdot[(u_x^{\delta,\nu})_+]_x \\ &\quad \leq -\frac{1}{\delta}\left[S_u(u_x^{\delta,\nu})_+ + S_v v_x^{\delta,\nu}\left(\frac{1+sgn(u_x^{\delta,\nu})}{2}\right)\right] + \nu[(u_x^{\delta,\nu})_+]_{xx},\end{aligned} \tag{3.4}$$

$$(v_x^{\delta,\nu})_t \leq \frac{1}{\delta}[S_u(u_x^{\delta,\nu})_+ + S_v v_x^{\delta,\nu}]. \tag{3.5}$$

By solving the second inequality, we find (with $S_v(\tau) := S_v(x,\tau) \equiv S_v(u^{\delta,\nu}(x,\tau), v^{\delta,\nu}(x,\tau))$ and $B(t) := \int_0^t S_v(\tau)d\tau$) that

$$v_x^{\delta,\nu}(t) \leq e^{\frac{B(t)}{\delta}} v_x^{\delta,\nu}(0) + \frac{1}{\delta}\int_0^t e^{\frac{B(t)-B(\tau)}{\delta}} S_u(\tau)(u_x^{\delta,\nu}(\tau))_+ d\tau. \tag{3.6}$$

Plugging this into (3.4) and denoting $m(t) = \max_x (u_x^{\delta,\nu}(x,t))_+$, we end up with

$$\dot{m}(t) \leq -\frac{S_u(t)}{\delta}m(t) - \frac{S_v(t)}{\delta}e^{\frac{B(t)}{\delta}}(v_x^{\delta,\nu}(0))_+ - \frac{S_v(t)}{\delta^2}\int_0^t e^{\frac{B(t)-B(\tau)}{\delta}} S_u(\tau)m(\tau)d\tau. \tag{3.7}$$

The first and the third terms in the right-hand side of (3.7) add up to a perfect derivative, modulo extra terms which are differentiated along the characteristics where $u_{xx}^{\delta,v}(x(t),t) = 0$ so that

$$\dot{m}(t) \leq \left(-e^{\frac{B(t)}{\delta}}(v_x^{\delta,\nu}(0))_+\right)_t - \frac{1}{\delta}\left(\int_0^t e^{\frac{B(t)-B(\tau)}{\delta}} S_u(\tau)m(\tau)d\tau\right)_t + k(t). \tag{3.8}$$

Here the constant $k(t)$ (depending on the convexity constant of F in (2.3), α) is an upperbound on the extra terms differentiated along the characteristics, e.g., $\partial_x B(x,t)\dot{x}e^{B(t)/\delta}(u_x^{\delta,v}(0))_+/\delta\ldots$. Integration of (3.8) over $(0,T)$ yields

$$m(T) \leq m(0) + (v_x^{\delta,\nu}(0))_+\left[1 - e^{\frac{B(T)}{\delta}}\right] - \frac{1}{\delta}\int_0^T e^{\frac{B(T)-B(\tau)}{\delta}} S_u(\tau)m(\tau)d\tau + \int_0^T k(\tau)d\tau. \tag{3.9}$$

In view of the positivity of S_u, we obtain that

$$(u_x^{\delta,\nu}(x,T))_+ \leq (u_x^{\delta,\nu}(x,0))_+ + (v_x^{\delta,\nu}(x,0))_+ + K_T, \quad K_T = \int_0^T k(\tau)d\tau,$$

and the assertion follows with Const $= ||u^{\delta,\nu}(\cdot,0)||_{Lip^+(x)} + ||v^{\delta,\nu}(\cdot,0)||_{Lip^+(x)} + K_T$. ∎

We close this section by noting that the proof of Assertion 3.1 is based on the straightforward, formal maximum principle for the positive part of $u^{\delta,\nu}$; alternatively, it could be justified, for example, by L^p iterations in (3.4).

4. *Lip*'-consistency and proof of the main result. In this section we prove the promised error estimate (2.4) in the Lip'-norm. According to the results of [Ta], [NT], the error $||u^{\delta,\nu} - \bar{u}||_{Lip'}$ is upper bounded by the truncation error

$$\left|\left|[u^{\delta,\nu} + v(u^{\delta,\nu})]_t + f(u^{\delta,\nu})_x\right|\right|_{Lip'(x,t)}. \tag{4.1}$$

This quantity measures by how much $u^{\delta,\nu}$ fails to satisfy the limiting equation (1.10). To complete this proof we have to show, therefore, that the truncation error is of order $\mathcal{O}(\epsilon(\delta) + \delta + \nu)$. We proceed as follows.

Adding the two components of the regularized system (1.5) to (1.4), we obtain that

$$u_t^{\delta,\nu} + v_t^{\delta,\nu} + f(u^{\delta,\nu})_x = \nu u_{xx}^{\delta,\nu},$$

which we rewrite as

$$\begin{aligned} *[u^{\delta,\nu} + v(u^{\delta,\nu})]_t + f(u^{\delta,\nu})_x &= u_t^{\delta,\nu} + v_t^{\delta,\nu} + f(u^{\delta,\nu})_x \\ +[v(u^{\delta,\nu}) - v^{\delta,\nu}]_t &= \nu u_{xx}^{\delta,\nu} + [v(u^{\delta,\nu}) - v^{\delta,\nu}]_t. \end{aligned}$$

It is here that we take advantage of the *weak* Lip'-norm introduced earlier in section 2: by measuring the L^1-size of its primitive, the right-hand side of the last equality tells us that the truncation error in (4.1) does not exceed

$$\begin{aligned} &||\nu u_{xx}^{\delta,\nu} + [v(u^{\delta,\nu}) - v^{\delta,\nu}]_t||_{Lip'(x,t)} \\ &\qquad \leq \text{Const}_T \cdot \Big[\nu ||u_x^{\delta,\nu}||_{L^1(x,t)} + ||v(u^{\delta,\nu}) - v^{\delta,\nu}||_{L^1(x,t)}\Big] \\ &\qquad =: \text{Const}_T \cdot \Big[\ I + II\ \Big]. \end{aligned} \tag{4.2}$$

We proceed with estimating the two terms on the right. First, since $u^{\delta,\nu}$ is Lip^+-bounded, (3.1), it has a bounded variation, $||u_x^{\delta,\nu}||_{L^1(x,t)} \leq C_K$ (where C_K may depend on the Lip^+-bound, K, and the finite support of $u^{\delta,\nu}$) and, therefore, $I \leq \mathcal{O}(\nu)$. Next, we find that the second term, II, is of order

$$II \equiv ||v(u^{\delta,\nu}) - v^{\delta,\nu}||_{L^1(x,t)} \sim ||S(u^{\delta,\nu}, v^{\delta,\nu})||_{L^1(x,t)}. \tag{4.3}$$

Indeed, since $0 < \eta \leq -S_v \leq$ Const, we have

$$\frac{1}{\eta} \leq \frac{|v(u^{\delta,\nu}) - v^{\delta,\nu}|}{|S(u^{\delta,\nu}, v(u^{\delta,\nu})) - S(u^{\delta,\nu}, v^{\delta,\nu})|} \leq \text{Const}$$

and, hence, $|v(u^{\delta,\nu})-v^{\delta,\nu}| \sim |S(u^{\delta,\nu},v(u^{\delta,\nu}))-S(u^{\delta,\nu},v^{\delta,\nu})| = |S(u^{\delta,\nu},v^{\delta,\nu})|$, and (4.3) follows. Returning to (4.2) we find that

$$
\begin{aligned}
||\nu u^{\delta,\nu}_{xx}+[v(u^{\delta,\nu})-v^{\delta,\nu}]_t||_{Lip'(x,t)} &\leq \text{Const}_T\cdot\Big[\ I+II\ \Big] \\
&\leq \text{Const}_T\cdot\Big[\nu+||S(u^{\delta,\nu},v^{\delta,\nu})||_{L^1(x,t)}\Big]. \qquad (4.4)
\end{aligned}
$$

To conclude with the promised $\mathcal{O}(\epsilon(\delta)+\delta+\nu)$-bound, it remains to prove that $||S(u^{\delta,\nu},v^{\delta,\nu})||_{L^1(x,t)}$—or, utilizing (4.3), that $\delta||v^{\delta,\nu}_t(\cdot,t)||_{L^1(x)}$ is of order $\mathcal{O}(\epsilon(\delta)+\delta)$,

$$
||S(u^{\delta,\nu}(\cdot,t),v^{\delta,\nu}(\cdot,t))||_{L^1(x)} \equiv \delta||v^{\delta,\nu}_t(\cdot,t)||_{L^1(x)} = \mathcal{O}(\epsilon(\delta)+\delta). \qquad (4.5)
$$

To achieve such an estimate, we differentiate (1.4) with respect to t, multiply by $sgn(u^{\delta,\nu}_t)$, and obtain

$$
\begin{aligned}
|u^{\delta,\nu}_t|_t+(f'(u^{\delta,\nu})u^{\delta,\nu}_t)_x sgn(u^{\delta,\nu}_t) = -\frac{1}{\delta}\Big(S_u|u^{\delta,\nu}_t|+S_v|v^{\delta,\nu}_t|sgn(u^{\delta,\nu}_t)sgn(v^{\delta,\nu}_t)\Big)& \\
+\epsilon(u^{\delta,\nu}_t)_{xx}sgn(u^{\delta,\nu}_t).& \qquad (4.6)
\end{aligned}
$$

The same treatment of equation (1.5) yields

$$
|v^{\delta,\nu}_t|_t = \frac{1}{\delta}\Big(S_u|u^{\delta,\nu}_t|sgn(u^{\delta,\nu}_t)sgn(v^{\delta,\nu}_t)+S_v|v^{\delta,\nu}_t|\Big). \qquad (4.7)
$$

Next, we integrate the following equations with respect to x:

$$
\frac{d}{dt}||u^{\delta,\nu}_t||_{L^1(x)} \leq -\frac{1}{\delta}\Bigg(\int_x S_u|u^{\delta,\nu}_t|dx+\int_x S_v|v^{\delta,\nu}_t|sgn(u^{\delta,\nu}_t)sgn(v^{\delta,\nu}_t)dx\Bigg), \qquad (4.8)
$$

$$
\frac{d}{dt}||v^{\delta,\nu}_t||_{L^1(x)} \leq \frac{1}{\delta}\Bigg(\int_x S_u|u^{\delta,\nu}_t|sgn(u^{\delta,\nu}_t)sgn(v^{\delta,\nu}_t)dx+\int_x S_v|v^{\delta,\nu}_t|dx\Bigg). \qquad (4.9)
$$

Finally, we add up (4.8) and (4.9), obtaining

$$
\begin{aligned}
\frac{d}{dt}\Big[||u^{\delta,\nu}_t||_{L^1(x)}+||v^{\delta,\nu}_t||_{L^1(x)}\Big] \leq \frac{1}{\delta}\Bigg[&\int_x S_u|u^{\delta,\nu}_t|\Big(sgn(u^{\delta,\nu}_t)sgn(v^{\delta,\nu}_t)-1\Big)dx \\
&+\int_x S_v|v^{\delta,\nu}_t|\Big(1-sgn(u^{\delta,\nu}_t)sgn(v^{\delta,\nu}_t)\Big)dx\Bigg] \leq 0.
\end{aligned}
$$

It follows that

$$
||u^{\delta,\nu}_t(\cdot,t)||_{L^1(x)}+||v^{\delta,\nu}_t(\cdot,t)||_{L^1(x)} \leq ||u^{\delta,\nu}_t(\cdot,0)||_{L^1(x)}+||v^{\delta,\nu}_t(\cdot,0)||_{L^1(x)} \qquad (4.10)
$$

and, in particular,

$$
\delta||v^{\delta,\nu}_t(\cdot,t)||_{L^1(x)} \leq \delta||u^{\delta,\nu}_t(\cdot,0)||_{L^1(x)}+\delta||v^{\delta,\nu}_t(\cdot,0)||_{L^1(x)}.
$$

To conclude this proof, we show that the upper bound on the right does not exceed the promised $\mathcal{O}(\epsilon(\delta)+\delta)$. Indeed, by equations (1.4)–(1.5), $u^{\delta,\nu}_t = -v^{\delta,\nu}_t-f(u^{\delta,\nu})_x+\nu u^{\delta,\nu}_{xx}$, and hence

$$
\begin{aligned}
\delta||u^{\delta,\nu}_t(\cdot,0)||_{L^1(x)}+\delta||v^{\delta,\nu}_t(\cdot,0)||_{L^1(x)} \leq\ & 2||S(u^{\delta,\nu}(\cdot,0),v^{\delta,\nu}(\cdot,0))||_{L^1(x)} \\
&+\delta||f(u^{\delta,\nu}(\cdot,0))_x||_{L^1(x)}+\delta\nu||u^{\delta,\nu}_{xx}(\cdot,0)||_{L^1(x)}.
\end{aligned}
$$

The three terms on the right are upper-bounded by $\mathcal{O}(\epsilon(\delta)+\delta)$ since, by our assumption of the "prepared" initial data (1.12), $||S(u^{\delta,\nu}(\cdot,0),v^{\delta,\nu}(\cdot,0))||_{L^1(x)} = \mathcal{O}(\epsilon(\delta))$; the BV-boundedness of $u^{\delta,\nu}$ yields $\delta||f(u^{\delta,\nu}(\cdot,0))_x||_{L^1(x)} = \mathcal{O}(\delta)$ and, finally, since the initial data are assumed to be in $W^2(L^1)$, then $\delta\nu||u^{\delta,\nu}_{xx}(\cdot,0)||_{L^1(x)} = \mathcal{O}(\delta\nu) << \mathcal{O}(\delta)$. This completes the proof of Theorem 2.1. $\square$

Remark. We close by noting that the $W^2(L^1)$-regularity of initial data used in the last stage of the proof can be relaxed. In fact, it is sufficient to assume $||u_{0x}||_{L^1} + \nu||u_{0xx}||_{L^1} \leq \text{Const.}$

Note added in proof. We thank Professor R. Natalini for pointing out a gap in the previous version of the proof of Assertion 3.1. Details will appear elsewhere.

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